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# 7.2 Areas in the Plane

## Area Between Curves

1. We partition the region into vertical strips of equal width $\Delta x$ and approximate each strip with a rectangle with base parallel to $[a, b]$ (Figure 7.4). Each rectangle has area

$$[f(c_k) - g(c_k)]\,\Delta x$$

for some $c_k$ in its respective subinterval (Figure 7.5). This expression will be nonnegative even if the region lies below the $x$-axis. We approximate the area of the region with the Riemann sum

$$\sum [f(c_k) - g(c_k)]\,\Delta x.$$

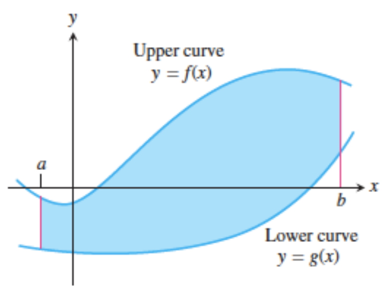


**Figure 7.3** The region between $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$.

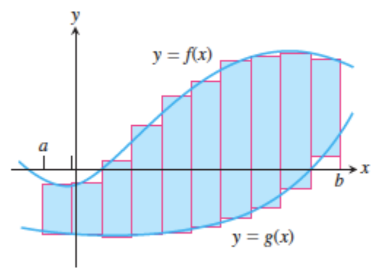


**Figure 7.4** We approximate the region with rectangles perpendicular to the $x$-axis.

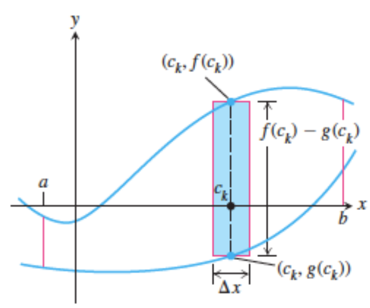


**Figure 7.5** The area of a typical rectangle is $[f(c_k) - g(c_k)]\,\Delta x$.

2. The limit of these sums as $\Delta x \to 0$ is

$$\int_a^b [f(x) - g(x)]\,dx.$$

This approach to finding area captures the properties of area, so it can serve as a definition.

**DEFINITION Area Between Curves**

If $f$ and $g$ are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area between the curves $y = f(x)$ and $y = g(x)$ from $a$ to $b$** is the integral of $[f - g]$ from $a$ to $b$,

$$A = \int_a^b [f(x) - g(x)]\,dx.$$

## EXAMPLE 1 Applying the Definition

Find the area of the region between $y = \sec^2 x$ and $y = \sin x$ from $x = 0$ to $x = \pi/4$.

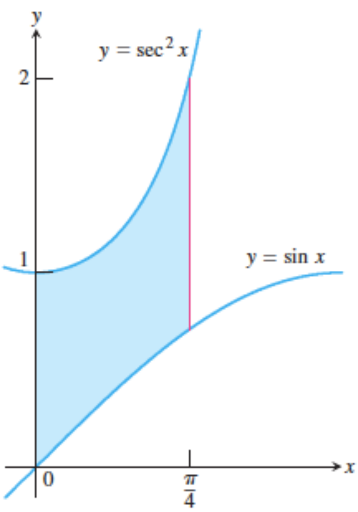


**Figure 7.6** The region in Example 1.

$$\int_0^{\pi/4} \left[ (\sec^2 x) - (\sin x) \right] dx$$

$$\tan x + \cos x \Big|_0^{\pi/4}$$

$$\left(1 + \frac{\sqrt{2}}{2}\right) - (0 + 1)$$

$$\frac{\sqrt{2}}{2}$$

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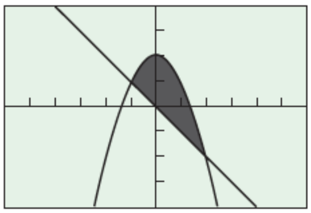
## Area Enclosed by Intersecting Curves

When a region is enclosed by intersecting curves, the intersection points give the limits of integration.

### EXAMPLE 2 Area of an Enclosed Region

Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

$y_1 = 2 - x^2$
$y_2 = -x$



[–6, 6] by [–4, 4]

**Figure 7.8** The region in Example 2.

limits of integration:

$2 - x^2 = -x$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = -1 \quad x = 2$

$$\int_{-1}^{2} \left[(2 - x^2) - (-x)\right] dx$$

$$\int_{-1}^{2} (-x^2 + x + 2)\, dx$$

$$= -\frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x \Big|_{-1}^{2}$$

$$\left(-\frac{8}{3} + 2 + 4\right) - \left(\frac{1}{3} + \frac{1}{2} - 2\right)$$

$$\left(-\frac{8}{3} + \frac{6}{3} + \frac{12}{3}\right) - \left(\frac{2}{6} + \frac{3}{6} - \frac{12}{6}\right)$$

$$\frac{10}{3} - \left(-\frac{7}{6}\right)$$

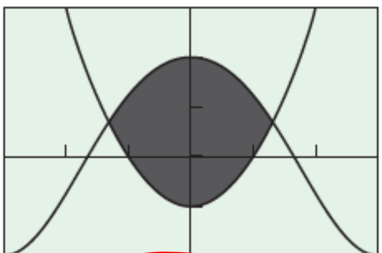
$$\frac{20}{6} + \frac{7}{6} = \frac{27}{6} = \frac{9}{2}$$

## EXAMPLE 3 Using a Calculator

Find the area of the region enclosed by the graphs of $y = 2\cos x$ and $y = x^2 - 1$.

$y_1$ $y_2$

$y_1 = 2\cos x$
$y_2 = x^2 - 1$



[–3, 3] by [–2, 3]

**Figure 7.9** The region in Example 3.

calculate and store the intersections

$A \approx -1.265423706$

$B \approx 1.265423706$

$$\int_A^B \left[(2\cos x) - (x^2 - 1)\right] dx$$

$\approx 4.995$

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## Boundaries with Changing Functions

If a boundary of a region is defined by more than one function, we can partition the region into subregions that correspond to the function changes and proceed as usual.

### EXAMPLE 4 Finding Area Using Subregions

Find the area of the region $R$ in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the $x$-axis and the line $y = x - 2$.

$$\int_2^4 (\sqrt{x} - x + 2)\,dx$$

$$\frac{2}{3}x^{3/2} - \frac{1}{2}x^2 + 2x \Big|_2^4$$

$$\left(\frac{2}{3}(8) - \frac{1}{2}(16) + 8\right)$$

$$- \cancel{\frac{2}{3}(2)^{3/2}} + \frac{1}{2}(4) - 4$$

The region is shown in Figure 7.10.

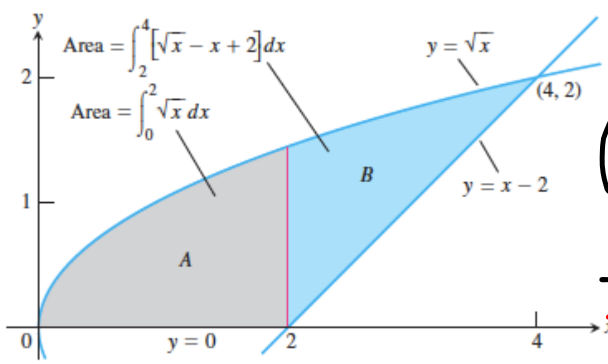


**Figure 7.10** Region $R$ split into subregions $A$ and $B$. (Example 4)

$$\int_0^2 \sqrt{x}\,dx$$

$$\frac{2}{3}x^{3/2} \Big|_0^2$$

$$= \cancel{\frac{2}{3}(2)^{3/2}}$$

$$= \frac{10}{3}$$

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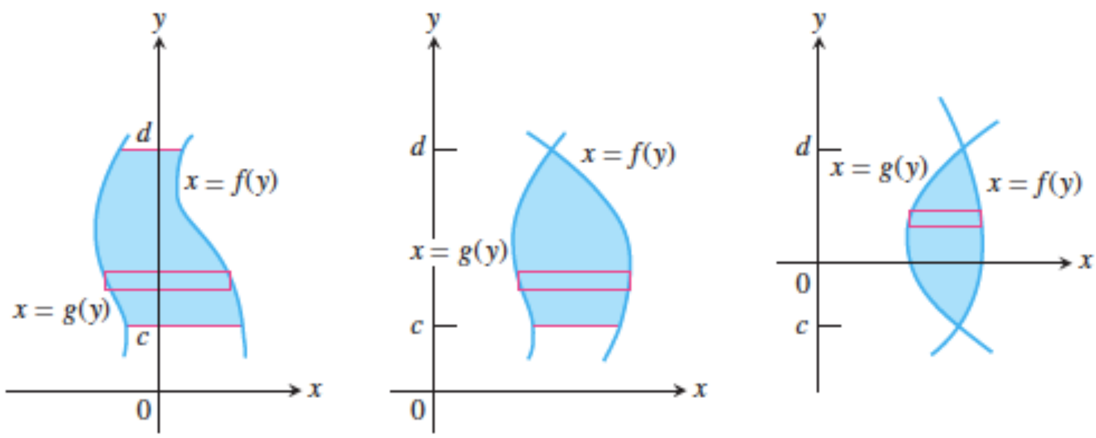
## Integrating with Respect to $y$

For regions like these




use this formula

$$A = \int_c^d [f(y) - g(y)]\,dy.$$

## EXAMPLE 5 Integrating with Respect to $y$

Find the area of the region in Example 4 by integrating with respect to $y$.

### EXAMPLE 4 Finding Area Using Subregions

Find the area of the region $R$ in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the $x$-axis and the line $y = x - 2$.

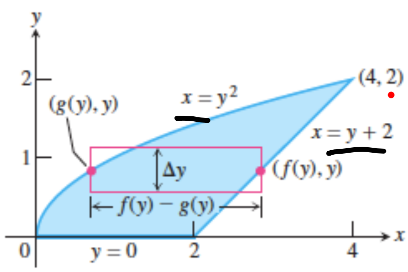



**Figure 7.11** It takes two integrations to find the area of this region if we integrate with respect to $x$. It takes only one if we integrate with respect to $y$. (Example 5)

$$y = \sqrt{x}$$

$$x = y^2$$

$$y = x - 2$$

$$x = y + 2$$

$$\int_0^2 (y + 2 - y^2)\, dy$$

$$-\frac{1}{3}y^3 + \frac{1}{2}y^2 + 2y \Big|_0^2$$

$$-\frac{1}{3}(8) + \frac{1}{2}(4) + 4$$

$$= \frac{10}{3}$$

### EXAMPLE 6 Making the Choice

Find the area of the region enclosed by the graphs of $y = x^3$ and $x = y^2 - 2$.

$x = y^{1/3}$

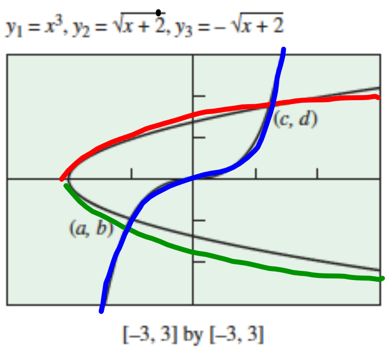



**Figure 7.12** The region in Example 6.

$$\int_0^d \left(y^{1/3} - y^2 + 2\right) dy$$

$\approx 4.215$

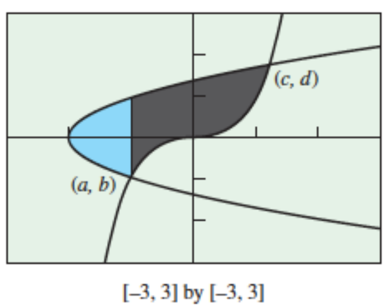



**Figure 7.13** If we integrate with respect to $x$ in Example 6, we must split the region at $x = a$.

$b = -1$

$d \approx 1.753003715$

## Saving Time with Geometry Formulas

Here is yet another way to handle Example 4.

### EXAMPLE 7 Using Geometry

Find the area of the region in Example 4 by subtracting the area of the triangular region from the area under the square root curve.

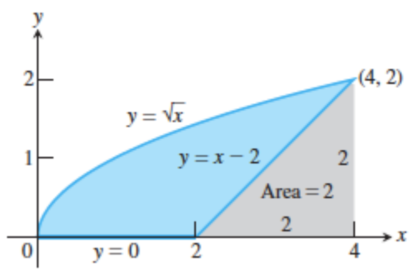



**Figure 7.14** The area of the blue region is the area under the parabola $y = \sqrt{x}$ minus the area of the triangle. (Example 7)

$$\int_0^2 \sqrt{x}\, dx + \text{area } \Delta$$

$$\tfrac{1}{2}(2 \cdot 2)$$

$$= \frac{10}{3}$$